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Criteria & Problem on Truncation of Taylor Expansions(C^∞ - maps and their Singularities)

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CITATION:

Kuo, Tzee-Char. Criteria & Problem on Truncation of Taylor Expansions(C^∞ - maps and their Singularities). 数理解析研究所講究録 1985, 550: 123-125

ISSUE DATE:

1985-02

URL:

<http://hdl.handle.net/2433/98884>

RIGHT:

Criteria & Problems on Truncation of Taylor Expansions

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Consider an analytic, or C^∞ , function germ $f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$. Let $T^\infty(f)$ denote its Taylor expansion

$$T^\infty(f) = \sum a_i x_i + \sum a_{ij} x_i x_j + \cdots, \quad a_i = \frac{\partial f}{\partial x_i}(0), \text{ etc.}$$

Recall that if some $a_i \neq 0$, then f is equivalent to x_i under a coordinate transformation (Implicit Function Theorem), and if all $a_i = 0$ but $\det(a_{ij}) \neq 0$, then f is equivalent to $\sum_{i=1}^n \pm x_i^2$ (Morse Lemma). In case all $a_i = 0$ and $\det(a_{ij}) = 0$, we like to consider the problem of finding the smallest integer r such that all terms of degree $\geq r+1$ in $T^\infty(f)$ can be omitted without effecting the local behavior of f . An answer to this problem is needed in calculus (Maximam & Minima), Differential Geometry (local theory), and for Singularity Theory (see Thom's Bombay Lecture [6]).

We recall some results and state some problems.

Call a polynomial $z(x_1, \dots, x_n)$, of degree r with $z(0)=0$, an r -jet. We say z is topologically determined (C^0 -sufficient) in C^r if for any C^r -function f with $T^r(f)=z$, there exists a local homeomorphism $h: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $f \circ h = z$. Here $T^r(f)$ denote the Taylor expansion of f up to degree r .

Theorem 1 ([1][3][4]) An r -jet z is topologically determined in C^r if and only if $\exists \epsilon > 0$,

$$|\text{Grad } z| \geq \epsilon |x|^{r-1}, \quad x \text{ near } 0.$$

Corollary. If $z = H_r(x_1, \dots, x_n)$ is a non-degenerate r -form, then $|\text{Grad } z| \geq \epsilon |x|^{r-1}$ and z is topologically determined as an r -jet.

Call an r -jet z C^1 -determined if for any f with $T^r(f) = z$, there exists a C^1 -local diffeomorphism h , $f \circ h = z$.

Problem 1. Consider the Koike-Kucharz function ([2])

$$k(x, y) = x^3 - 3xy^s, \quad s \geq 3.$$

Find the smallest r such that $T^r(k)$, as an r -jet, is C^1 -determined.

We believe a solution of this problem will lead to interesting criteria for C^1 -determinacy.

Theorem 1 is an analytic criterion. Next we recall a geometric criterion. Given an r -jet $z(x_1, \dots, x_n)$. Consider

$$F(x; \lambda) = z(x) + \sum_{|\alpha|=r} \lambda_\alpha x^\alpha, \quad \lambda_\alpha \text{ indeterminants,}$$

$\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}^+$, $\alpha_i \geq 0$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Let $V_F = \{(x, \lambda) \mid F(x; \lambda) = 0\}$. This is an algebraic variety in $\mathbb{R}^n \times \Lambda$, Λ the space of parameters $\lambda = (\lambda_\alpha)$. It is easy to see that the singular subvariety of V_F is $\{0\} \times \Lambda$.

Theorem 2 ([5]) z is topologically determined in C^r if and only if V_F is Whitney (a, b) -regular over Λ at 0 .

Problem 2. Translate the above two theorems into algebra, i.e. find algebraic criteria for topological determinacy.

Problem 3. Find geometric and analytic criteria for z to be blow-analytically determined.

References

- [1] J. Bochnak and S. Lojasiewicz, A converse of the Kuiper-Kuo Theorem. Proc. Liverpool Singularities Symposium 1, Lecture Notes in Math. No. 192. Springer-Verlag, 1971.
- [2] S. Koike and W. Kucharz, Sur les realisations de jets non-suffisants, C.R. Acad. Sc. Paris, 288, 457-459 (1979).
- [3] N.H. Kuiper, C^1 -equivalence of Functions near Isolated Critical Points, Symp. Inf. Dim. Top., Ann. Math. Studies 69, Princeton University Press 1972.
- [4] T.C. Kuo, On C^0 -sufficiency of jets of potential functions, Topology 8 (1969), 167-171.
- [5] T.C. Kuo and Y.C. Lu, Sufficiency of jets via stratification theory, Inventiones Math, Vol 57, (1980), 219-226.
- [6] R. Thom, Local topological properties of differentiable mappings, Differential Analysis, Bombay Colloquium 1964, Oxford University Press.